## Exam / Numerical Mathematics 1 / June 15th 2020, University of Groningen

A simple calculator is allowed.
You are allowed to use the material of the course (lecture notes, tutorials).
All answers need to be justified using mathematical arguments.
**Total time: 3 hour 30 minutes (time includes upload of the PDF with your answers to Nestor) + 30 minutes (if special needs)**
**Remember: oral "checks" may be run afterwards.**

### Exercise 1 (9 points)

Consider a function $f(x) \in C^6([0, 2h])$. We define the numerical integration rule:

$$\tilde{I}(f) = \frac{h}{3}\big(f(0) + 4f(h) + f(2h)\big)$$

(a) $\boxed{1.5}$ Show that

$$\tilde{I}(f) = 2hf(0) + 2h^2 f'(0) + \frac{4}{3}h^3 f''(0) + \frac{2}{3}h^4 f'''(0) + h^5\left(\frac{1}{18}f''''(\xi_1) + \frac{2}{9}f''''(\xi_2)\right)$$

with $\xi_1, \xi_2 \in (0, 2h)$.

(b) $\boxed{3}$ Show that

$$\tilde{I} - \int_0^{2h} f(u)du \leq 0.6h^5 \max_{\eta\in[0,2h]} |f''''(\eta)|$$

(c) $\boxed{2}$ Show that the degree of exactness of the numerical integration rule is 3.

(d) $\boxed{1}$ Is this integration formula more or less accurate than the Simpson's rule? Justify your answer.

(e) $\boxed{1.5}$ Extend the numerical integration rule to build an integration rule over an interval $[a, b]$ using two subintervals of length $\delta = (b - a)/2$. Give an error bound for this new rule.

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## Exercise 2 (9 points)

Consider the problem: Find $x^*$ such that $f(x^*) = 0$ solved via the Newton iterations

$$x^{(k+1)} = \phi(x^{(k)}) = x^{(k)} - f(x^{(k)})/f'(x^{(k)})$$

(a) $\boxed{2}$ Show that if $f(x) = (x - x^*)^m g(x)$, $m > 0$, $g(x^*) \neq 0$, then $\phi'(x^*) = 1 - 1/m$.

(b) $\boxed{1.5}$ Using Newton iterations for finding the root of $f(x) = x^3$ leads to the following iterands

| $k$ | $x^{(k)}$ |
|---|---|
| 0 | 15.0000 |
| 1 | 10.0000 |
| 2 | 6.6667 |
| 3 | 4.4444 |
| 4 | 2.9630 |
| 5 | 1.9753 |

Compute the convergence order of these iterations and explain the obtained value using the theory studied in the course.

(c) $\boxed{1}$ Propose a new iteration function $h(x)$, by modifying $\phi(x)$, such that $h'(x^*) = 0$

(d) $\boxed{1.5}$ Using the modified iteration function $h(x)$, perform two iterations to approximate the root of $x^3$, with $x^{(0)} = 15$. Explain the difference in the convergence order with respect to the standard Newton iterations.

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Consider solving the linear system of equations $Ax = b$, with $A \in \mathbb{R}^{n \times n}$, using the following iterative procedure

$$x_k = x_{k-1} + \alpha \left( b - Ax_{k-1} \right), \quad k \geq 1,$$

with

$$A = \begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(e) $\boxed{2}$ Is it possible to choose a constant value of $\alpha$ such that the Richardson iterations converge? Justify your answer just by recalling the proofs studied in the course.

(f) $\boxed{1}$ In case you answered "yes" in the previous question, give a value of $\alpha$ that makes the iterations converge. If you answered "no", re-write the linear system (such that the solution is the same) so you can pick a value of $\alpha$ for making the solutions converge.

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## Exercise 3 (9 points)

Consider the system of ODEs for the functions $d(t), v(t)$:

$$\begin{cases} mv' = -k\exp(d)d - cv & v(0) = v_0 \\ d' = v - \gamma d, & d(0) = d_0 \end{cases}, \tag{1}$$

with $k, c, \gamma > 0$.

(a) $\boxed{2}$ Show that:

$$(v^2)' + g(d)(d^2)' \leq 0$$

for some function $g(d) > 0$.

(b) $\boxed{1}$ Discretize () in time using the $\beta$-method, and denote the discrete solutions by $d_n \approx d(t_n), v_n \approx v(t_n)$. Formulate the system of (possibly non-linear) equations for $(d_{n+1}, v_{n+1})$ as a vector root finding problem $T(d_{n+1}, v_{n+1}) = 0$, and give the specific form for $T$. Assume $t_{n+1} = t_n + h$, $h > 0$.

(c) $\boxed{3}$ Write down the Newton iteration for computing the $(k+1)$-st iterand $d_{n+1}^{k+1}, v_{n+1}^{k+1}$ from the $k$-th iterand $d_{n+1}^{k}, v_{n+1}^{k}$. Give specific expressions for the vectors and matrix involved, but you do not need to explicitly invert any matrix.

(d) $\boxed{3}$ Consider now the system of ODEs:

$$X'(t) = AX(t), \quad X(0) = X_0 \neq 0, \tag{2}$$

with

$$A = \begin{bmatrix} -1 & -1/2 \\ 1/2 & -2 \end{bmatrix}.$$

Discretize the ODE ((d)) using the $\beta$-method for $\beta = 0$. Then, determine $h_{crit}$ such that $X_n \to 0$ (for any $X_0$) when $n \to \infty$ for $h < h_{crit}$.